Lecture 12: Dimension theorems and bases

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10. Dimension theorems

We know: the size of any linearly independent set in V is:

$\leq$ dim(V)

$\leq$ size of any spanning set of V

and that a basis is the biggest possible LI set in V and the smallest possible spanning set of V.

Obtaining bases

Recall:

1) Every linearly independent subset of V can be extended to a basis of V: $\{(1,0)\} \rightarrow \{(1,0),(1,1)\}$
2) Every spanning set of V can be reduced to a basis of V $\{(1,0),(0,1),(1,1)\} \rightarrow \{(1,0),(0,1)\}$

If we know that V is finite dimensional and we want to find a basis for a subspace U, we could either:

1) start taking non-zero vectors from U forming larger and larger LI sets or
2) start with a [finite] spanning set of U and cut it down

Theorem: Let V be a vector space with dim(V)=n<∞. Then:

1) any linearly independent set $\{v_1, \dots, v_n\}$ is a basis of V
2) any spanning set $\{v_1, \dots, v_n\}$ is a basis of V

PROOF

1) If $\{v_1, \dots, v_n\}$ didn't span V, pick v $\notin$ span$\{\{v_1, \dots, v_n\}$. Then $\{v, v_1, \dots, v_n\}$ is linearly independent.
   In contradiction to:
   $size\ of\ any\ linearly\ independent\ set\ in\ V \leq \dim(V)$
2) If $\{v_1, \dots, v_n\}$ was linearly dependent, remove one element v_k without changing the span. Then, $\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n\}$ is still spanning V.
   In contradiction to:
   $\dim(V) \leq$ size of any spanning set of V

Theorem: Let V be a vector space with dim(V)=n<∞. Let U be a subspace of V. This means:

1) 0$\leq \dim(U) \leq$ dim(V)
2) $\dim(U) = \dim(V)$ if and only if $I = V$
3) $\dim(U) = 0$ if and only if $u = \{0\}$

NOTE: If U is a subspace of V, dim(U)=m. Then, any subspace of V which is contained in U is also a subspace of U and, as such, has dimension $\leq$ m.

10.3 Examples

a) {(1,2),(3,4)} is linearly independent because none of the two vectors is a multiple of the other. But, $\dim(\mathbb{R}^2) = 2$. So, {(1,2),(3,4)} is a basis by the first theorem.
b) Any subspace U of $M_{22}(\mathbb{R})$ has 0$\leq \dim(U) \leq 4$ because $\dim\left(M_{22}(\mathbb{R})\right) = 4$. If dim(U)=4, then U is the full vector space $M_{22}(\mathbb{R})$.

10.4 Basis coordinates

basis $B_1 = \{(1,0),(0,1)\}$

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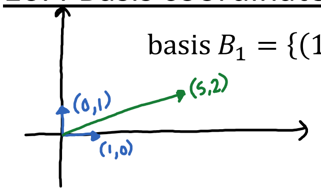
basis $B_1 = \{(1,0), (0,1)\}$

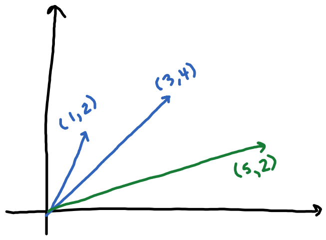


$(5,2) = 5(1,0) + 2(0,1) = [5,2]_{B1}$

"coordinates" → unique address using B_1

Another example

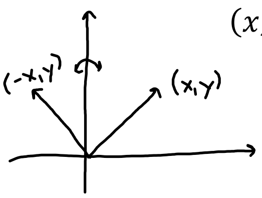
$B_2 = \{(1,2), (3,4)\}$
$(5,2), = -7(1,2) + 4(3,4)$
$= [-7,4]_{B2}$

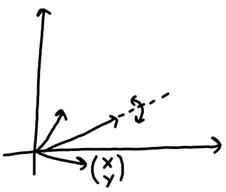


Why is this notation useful?

i) helps to understand complicated spaces
ie. $\mathbb{P}_2$
basis$= \{x^2, x, 1\} =: B$
$7x^2 - 3x + 5 = [7,-3,5]_B$ ⟵ identify ⟶ (7,-3,5)

ii) Choosing appropriate coordinate systems
reflection at y-axis
$(x,y) \to (-x,y)$





Reflection at axis spanned by (5,2)
$(x,y) \to ?$

We can solve this by choosing a new basis:
$B := \{(5,2), (2,-5)\}$
$[a,b]_B \Rightarrow [a,-b]_B$

10.5 Final theorem

How do we know that this address is unique?

Theorem

Let $B = \{v_1, \dots, v_n\}$ be an ordered basis for a vector space V. Then, for every $v \in$ V, there are unique $a_1, \dots, a_n \in \mathbb{R}$ such that:
$v = a_1v_1 + \cdots + a_nv_n = [a_1, \dots., a_n]_B$

PROOF

Existence of $a_1, \dots, a_n \in \mathbb{R}$ is clear because B spans V. Suppose there were 2 addresses $a_1, \dots, a_n \in \mathbb{R}$ and $b_1, \dots, b_n \in \mathbb{R}$.
Then:
$a_1v_1 + \cdots + a_nv_n = b_1v_1 + \cdots + b_nv_n$
$\Rightarrow (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n = 0$

$\Rightarrow \forall$ i : $a_i - b_i = 0$ (since B is linearly independent)
$\Rightarrow \forall$ i : $a_i = b_i$
Therefore this is unique.